ON THE PROBLEM OF TWO PISTONS

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A number of problems related to polytropic gas flows which arise when the walls of an infinite dihedral angle (planes P_1 and P_2) filled with gas which at the initial instant was at rest, begin to move out from the gas at constant velocities V_1 and V_2 was solved in paper [1]. The planes P_1 and P_2 – play the role of pistons moving parallel to their original positions. It was shown that when the withdrawal velocity of these planes is sufficiently high (in comparison with the velocity of sound c_0 in the gas at rest), then a zone of vacuum may arise at the edge of the dihedral angle (the piston intersection line). Solutions of these problems were derived from the domains of simple and double self-similar potential waves, and from the domains of constant motion. The subject of analysis in [1] was in essence limited to the consideration of occurance of the vacuum zone, while the case in which the latter is absent, and a separation free flow is realized was not examined.

In the following the analysis of the problem of two pistons moving out from a gas is largely completed with certain limitations as to the class of examined flows. It is shown that there exist three possible patterns of flows thus generated.

1. An isentropic separation free potential flow occurs throughout the region of perturbed motion with an area of constant flow at the dihedral angle edge which along a certain characteristic surface adjoins an area of unsteady potential double wave.

2. Strong discontinuities (shock waves) appear at the angle edge, the flow ceases to be potential, and it becomes necessary to apply the class of two-dimensional self-similar flows (with variables $\xi_i = x_i / t, | i = 1,2)$ with variable entropy.

3. A vacuum region appears at the angle edge. In the perturbed area the equation of double waves is always of the hyperbolic type, and the solution can be found by the method of characteristics [1].

The case of a high initial density gas (high velocity of sound c_0) and velocities V_1 and V_2 small as compared with c_0 is considered in this paper. It has been possible to derive here an analytical solution of the problem of separation free flow, and to define the values of angle α ($0 \leqslant \alpha \leqslant \pi / 2$, the angle between planes P_1 and P_2) for which a potential separation free flow may occur. The obtained simplified equation provides a suitable model for the analysis of flow patterns at low velocities V_1 and V_2.

In the case of the vacuum zone occurence the flow pattern singularities were investigated at the "vacuum–gas" boundary.

1. Conditions of the problem do not contain parameters of the length dimension, hence the solution, i. e. functions u_1, u_2 (velocity vector components) and the velocity of sound c will depend on two self-similar variables $\xi_1 = x_1 / t$ and $\xi_2 = x_2 / t$, where x_i are the Cartesian coordinates, and t is time. We write the system of equations defining the self-similar simple waves in the form [1 and 2]

$$u_1'^2 + u_2'^2 = 1 \qquad (u_i = u_i(\theta))$$

$$\xi_1 u_1' + \xi_2 u_2' - {}^1/_2 (\gamma - 1)\theta - u_1 u_1' - u_1 u_2' = 0 \qquad (\theta = 2(\gamma - 1)^{-1} c)$$

Here the prime denotes differentiation with respect to θ and γ is the adiabatic exponent in the equation of state

$$p = a^2 \rho^{\gamma} \qquad (a^2 = \text{const})$$

Here p is the pressure and ρ is the density.

For double potential waves we have [1 and 3]

$$\tfrac{1}{2}(\gamma - 1)\,\theta\,[(1 - \theta_1^2)\,\theta_{22} + 2\theta_1\theta_2\theta_{12} + (1 - \theta_2^2)\,\theta_{11}] + \\ + \tfrac{1}{2}(\gamma - 3)(\theta_1^2 + \theta_2^2) + 2 = 0 \quad (1.2)$$

$$\xi_i = u_i + \tfrac{1}{2}(\gamma - 1)\,\theta\theta_i \qquad \left(\theta_i = \frac{\partial\theta}{\partial u_i}, \quad \theta_{ik} = \frac{\partial^2\theta}{\partial u_i \partial u_k}; \quad (i = 1, 2)\right) \quad (1.3)$$

The analysis is carried out simultaneously in the hodograph plane of velocities u_1, u_2 (in which Eq. (1.2) is solved) and in the plane of self-similar variables ξ_1, ξ_2.

It was stated in [1] that at low velocities V_1 and V_2 when the flow pattern is free of separation, a line of parabolicity appears in the solution of Eq. (1.2) of double waves in the hodograph plane, beyond which there is an area of this equation ellipticity occupying a certain neighborhood of point

$$O'\ (-V_1,\ -(V_1 \cos\alpha + V_2)\,/\,\sin\alpha)$$

These results were obtained numerically. However more thorough investigations (in particular by increasing the number of computation points along each of the characteristics in order to decrease the computation error) had shown that the region of the hodograph plane bounded by a line along which $R^2 = \theta_1^2 + \theta_2^2 - 1 = 0$ is so small that it may be taken as being the point O'. The region corresponding to it in the $\xi_1\xi_2$ plane remains nevertheless fairly large. Calculations had shown that the characteristics of Eq. (1.2) tend to converge at point O', and that the values of functions θ_i at point O' depend on the direction of approach to that point, i.e. functions θ_1 and θ_2 at point O' are multiple-valued [4].

It may therefore be assumed that in the case of occurance of a separation free flow the double wave type of flow adjoins along a certain line in the $\xi_1\xi_2$ plane a flow with constant parameters; this corresponds to point O' in the hodograph plane.

The problem of junction of the double wave type of flow to the region at rest was investigated in paper [5] in which the singularities of solutions of Eq. (1.2) in the vicinity of the junction line were also explained. The results obtained in [5] may be readily extended to the case of a double potential wave adjoining a constant flow.

In order to establish the multivalence of θ_1 and θ_2 at point O', we pass over in Eq. (1.2) to new variables, putting

$$u_1 = \sigma \cos\varphi - V_1, \qquad u_2 = \delta \sin\varphi - (V_1 \cos\alpha + V_2)\,/\,\sin\alpha \quad (1.4)$$

In the new coordinates the line segment $\delta = 0$ along which $\theta_\varphi = 0$, $\theta_\delta = 1$ corresponds to point O', and the system of equations of the double wave takes the form

$$\frac{\gamma - 1}{2}\,\theta\left[\theta_{\delta\delta}\left(1 - \frac{\theta_\varphi^2}{\delta^2}\right) + \frac{1 - \theta_\delta^2}{\delta^2}\,\theta_{\varphi\varphi} + 2\frac{\theta_\delta\theta_\varphi}{\delta^2}\,\theta_{\delta\varphi} + \right. \\ \left. + \frac{\theta_\delta}{\delta}(1 - \theta_\delta^2) - 2\frac{\theta_\delta\theta_\varphi^2}{\delta^3}\right] + \frac{\gamma - 3}{2}\left(\theta_\delta^2 + \frac{\theta_\varphi^2}{\delta^2}\right) + 2 = 0 \quad (1.5)$$

$$\xi_1 = \delta\cos\varphi - V_1 + \frac{\gamma - 1}{2}\,\theta\left(\theta_\delta\cos\varphi - \theta_\varphi\frac{\sin\varphi}{\delta}\right)$$

$$\xi_2 = \delta\sin\varphi - \frac{V_1\cos\alpha + V_2}{\sin\alpha} + \frac{\gamma - 1}{2}\,\theta\left(\theta_\delta\sin\varphi + \theta_\varphi\frac{\cos\varphi}{\delta}\right) \quad (1.6)$$

two methods whenever a separation free flow pattern was realized.

The pattern of separation free flow for the case of $\gamma = 3$, $\alpha = {}^1/_2\pi$, $V_1 = V_2 = 0.4$ is shown on Fig. 1 and 2 in the u_1u_2 and the $\xi_1\xi_2$ planes respectively. The notations on Fig. 2 are: – 1 for regions of constant flow, 2 for regions of simple waves, and 3 for regions of double waves. On Fig. 1 lines $D'P'$, $D_1'P'$, $E'D'$, $E'D_1'$, $P'R'$, $P'R_1'$, $R'L'$, $R_1'L'$, $L'G'$, $L'G_1'$ correspond to regions denoted by 2, and points P', R', R_1', L', O' to those denoted by 1. Thus for example, region $DCNP$ corresponds to line $D'P'$, and region $PNKN_1$ to point P'. (Points on Fig. 1 are denoted by the same letters as on Fig. 2, but with a prime).

Fig. 3 shows the behavior of characteristics (in δ, φ coordinates) derived by the method of numerical integration of Eq. (1.2) in region $L'G'O'G_1'$ (Fig. 1). (The scale along the $O\delta$-axis has been increased tenfold). Characteristics (1.8) drawn through point L' (lines $L'G''$ and $L'G_1''$) are also shown for the sake of comparison.

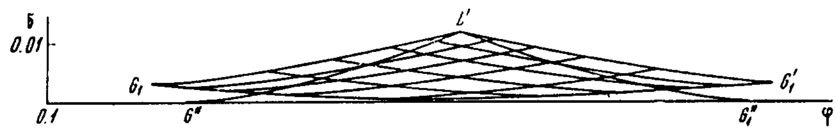

Fig. 3

Note 1.1. It is not possible to effect a direct junction of the constant motion to the double wave right at the beginning avoiding the construction of simple waves. In order to realize such a junction it would be necessary for a parabolic degeneration of Eq. (1.2) to be present and, furthermore, in accordance with the results of [5] the density of the double wave type of flow must increase with increasing distance from the contiguity line. If (1.2) in the neighbornhood of the degeneration line is of the hyperbolic type, then it is impossible to satisfy these two conditions right at the beginning.

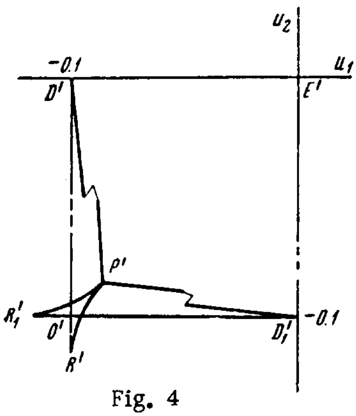

Fig. 4

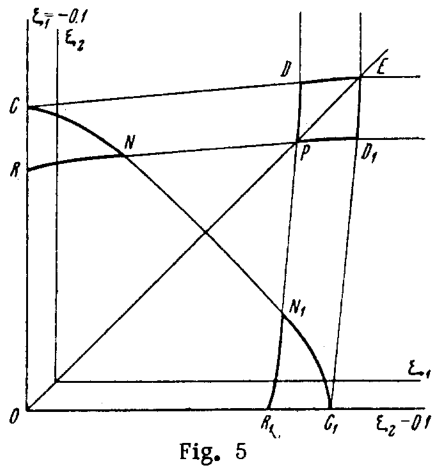

Fig. 5

2. Numerical calculations have shown that the occurance of separation free flows at low velocitites V_1 and V_2 examples of which were adduced in the preceding Section is very rare, and obtains for certain specific relationships between angle α and velocities V_1, V_2 only. In the derivation of the specific algorithm for the construction of flows at low V_1 and V_2 we obtain, as a rule, a nonunique congruence of sets of pairs (ξ_1, ξ_2) and

(u_1, u_2) corresponding to the flow and, moreover, the characteristics extend in the hodograph plane beyond the natural region of flow definition (e.g. beyond the rectangle $-V_1 \leqslant u_1 \leqslant 0$, $-V_2 < u_2 < 0$ when $\alpha = \pi/2$). This fact is not accidental, and is not related to errors of numerical calculations.

Let us consider the particular case of α and γ bound by the relationship

$$\frac{+1}{\gamma - 3} + \operatorname{ctg}^2 \frac{\alpha}{2} = 0, \qquad 1 < \gamma < 3, \quad 0 < \alpha < \frac{\pi}{2} \tag{2.1}$$

(the case of the so-called compatible α and γ [4]).

We shall give a strict proof of the occurance at specific values of V of the superposition in the hodograph plane of regions corresponding to different regions in the $\xi_1\xi_2$ plane $(V_1=V_2=V)$.

Fig. 4 and 5 show respectively in the u_1u_2 and the $\xi_1\xi_2$ planes a solution obtained numerically for $\alpha = {}^1/_2\pi$, $\gamma = 3$, $V = 0.1$, $\theta_0 = 1$ with superposition of regions in the hodograph plane. We shall use the notations of Fig. 4 and 5, and assume $0 < \alpha < {}^1/_2\pi$ (for such α the pattern remains qualitatively valid).

In the case of (2.1) the equation of the simple and double wave contiguity curve ED is of the form [1]

$$\xi_1 = \frac{\gamma+1}{2}\theta - \theta_0, \qquad \xi_2 = \frac{\gamma^2 - 1}{2(3-\gamma)}\theta \tag{2.2}$$

i.e. ED is a straight line along which $\theta_1 = 1$, $\theta_2 = \operatorname{ctg}{}^1/_2\alpha$.

Similar formulas hold for curve ED_1, i.e. the solution of Eq. (1.2) in area $EDPD_1$ may be written in the form

$$\theta = u_1 + \operatorname{ctg}{}^1/_2\alpha u_2 + \theta_0 \tag{2.3}$$

Riemann waves of the form

$$u_1 = -\cos\alpha\,(\theta - \theta_0 + V \operatorname{tg}\alpha \operatorname{ctg}{}^1/_2\alpha), \qquad u_2 = \sin\alpha\,(\theta - \theta_0 + V) \tag{2.4}$$

adjoin area $EDPD_1$ through the characteristic DP, and correspondingly

$$u_1 = \theta - \theta_0 + V \sin\alpha \operatorname{ctg}{}^1/_2\alpha, \qquad u_2 = -V\sin\alpha \tag{2.5}$$

through the characteristic D_1P.

From (2.4), (2.5) we have $\theta = \theta_0 - 2V$ at point P', i.e. a vacuum zone appears in the region $EDPD_1$, when $V \geqslant \theta_0/2$. Let $V < \theta_0/2$. We shall determine the velocity V at which a superposition of regions $D'P'R'$ and $D_1'P'R_1'$ will occur. The slope of the bisectrix $O'E'$ is equal to $\operatorname{ctg}{}^1/_2\alpha$, and the corresponding coefficient for the characteristic $P'R'$ at point P' is found from the relationship

$$\left(\frac{du_2}{du_1}\right)_{P'} = \frac{\theta_1\theta_2 - \sqrt{\theta_1^2 + \theta_2^2 - 1}}{1 - \theta_2^2}$$

where θ_i are to be taken at point N.

For the characteristic $P'R'$ to lie below $O'E'$, i.e. for the occurrence of a superposition of regions $D'P'R'$ and $D_1'P'R_1'$ the inequality

$$\left(\frac{du_2}{du_1}\right)_{P'} - \operatorname{ctg}\frac{\alpha}{2} = b \operatorname{ctg}\frac{\alpha}{2}\left[-\cos\alpha\left(b + \frac{2a}{\cos\alpha\cos{}^1/_2\alpha}\right)\right]^{-1} > 0$$

$$a = \left[\cos^2\frac{\alpha}{2} - \frac{\gamma}{\gamma+1}\left(\frac{\theta_0 - 2V}{\theta_0 - V}\right)^{(3-\gamma)/(\gamma-1)}\right]^{1/2}, \qquad b = a^2\sin^2\frac{\alpha}{2} + \frac{2a}{\cos{}^1/_2\alpha} - 1 \tag{2.6}$$

must be fulfilled.

By virtue of (2.1)

$$b + \frac{2a}{\cos\alpha\cos{}^1/_2\alpha} = \frac{(\gamma-1)^2}{(\gamma+1)(3-\gamma)} + 1 > 0$$

In order to satisfy inequality (2.6) it is necessary for $b < 0$ which is possible when

$1 < \gamma < 2$, i.e. when $1/_3\pi < \alpha < 1/_2\pi$. In this case the piston withdrawal velocity satisfies inequality

$$V < \theta_0 \frac{1-B}{2-B} = V^* \qquad \left(B = \left[\frac{3\gamma - 5 + (3-\gamma)^{1/2}}{\gamma}\right]^{(\gamma-1)/(3-\gamma)}\right) \tag{2.7}$$

in particular for $\gamma = 1{,}4$ $\alpha = \arccos 0.2$, $V^* \approx 0.03\ \theta_0$.

Thus, if inequality (2.7) is fulfilled, the mapping of the neighborhood of point P' of the hodograph plane onto the physical space is nonunique.

It is shown on Fig. 4 that in the numerical solution of the mixed problem of Eq. (1.2) point R' of characteristic $P'R'$ in the region $D'P'R'$ moves beyond the natural domain of flow determination, i.e. the square $O'D'E'D_1'$. The velocity vector component u_2 along the piston wall CO (Fig. 5) cannot be in this case monotonic (at point R $u_2 < -0.1$, hence a compression wave must appear along CO, and the flow ceases to be potential and isentropic.

Note 2.1. If one considers the problem of piston withdrawal at arbitrarily variable velocities (with zero acceleration at $t = 0$) and looks for a solution in the class of non-self-similar double waves, one finds as a rule that it is impossible to derive for this class a solution free of singularities. Apparently, as in the self-similar case weak shock waves may form at the angle vertex.

Note 2.2. It is always possible to derive a solution in the class of self-similar double potential waves when the angle between the withdrawing pistons is $\alpha > 1/_2\pi$, but only with sufficiently high velocities V_1 and V_2 when there is an outflow of gas into vacuum. In the case of low V_i however it is generally impossible to find in this class a solution free of singularities. (In particular, when one of the planes remains stationary, a weak shock wave is bound to appear in the neighborhood of the dihedral angle edge).

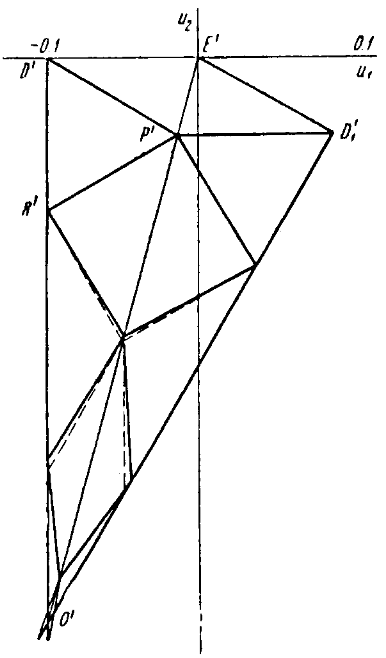

Fig. 6

3. The majority of results adduced above were obtained by solving Eq. (2.1) numerically. The derivation of analytical solutions of Eq. (2.1) even for certain regions only is very difficult due to its nonlinearity. It is of interest to attempt a simplification of this equation by imposing certain limitations on the physical content of the problem so as to obtain a simplified equation which would permit an analytical solution of the Goursat and of mixed problems with boundary conditions appropriate to the problem of two pistons considered. It appears that this can be done in the case of a "heavy" gas and low V_1 and V_2 when a vacuum zone is not generated. In that case the basic qualitative properties of separation free flows enumerated in Sections 1 and 2 are retained, while the quantitative results derived with the aid of the simplified equation are in good agreement with those obtained from Eq. (1.2) by the method of characteristics (see, e.g. Fig. 6 where the simplified solution is shown by dotted lines, and that of (1.2) by solid lines).

We shall consider a "heavy" gas of high initial density $\theta_0 \gg 1$ at $V_1 \ll \theta_0$, $V_2 \ll \theta_0$ under the conditions of our problem. Variations of the velocity of sound throughout the flow area will not be great with low V_1 and V_2, and it may be assumed that everywhere $\theta \gg 1$. As the region of variation of ξ_1, ξ_2 is bounded, it may be assumed in accordance with (1.3) that θ_1, θ_2 are bounded, and the magnitude

$$\eta = [{}^1/_2 (\gamma - 3) (\theta_1^2 + \theta_2^2) + 2] \,/\, {}^1/_2 (\gamma - 1) \theta$$

is small throughout the flow area.

Hence the term η in (1.2) after the latter had been divided by $(\gamma - 1)\theta / 2$ may be neglected, and the simplified equation is then reduced to the form

$$(1 - \theta_1^2)\, \theta_{22} + 2\theta_1\theta_2\theta_{12} + (1 - \theta_2^2)\theta_{11} = 0 \qquad (3.1)$$

Equation (3.1) has solutions of the form

$$\theta = a_1 u_1 + a_2 u_2 + a_3$$

where a_i are arbitrary constants. It appears that all necessary solutions of the mixed and of the Goursat problems are of this form.

In the following those regions of the hodograph plane in which the Goursat problem is solved will be called regions of the G type, and those containing mixed problem solutions of the S type (e.g. on Fig. 6 region $E'D'P'D_1'$ is of the G type, while region $D'P'R'$ is of the S type).

Theorem. If the angle between pistons P_1 and P_2 is $\alpha \in (0, {}^1/_2\pi)$ (piston P_1 corresponds to $\xi_1 = -V_1$), V_1, $V_2 \ll \theta_0$, then a separation free flow without singularities may be derived with the aid of Eq. (3.1) (in the class of entropic potential flows) only for

$$\alpha = \pi / k; \qquad k = 1, 2, 3, \ldots \qquad (3.2)$$

For odd k the flow region becomes closed following solutions ${}^1/_2 (k - 1)$ of G type problems, and $(k - 1)$ of problems of the S type, while for even k it is closed after solutions ${}^1/_2 k$ of G type problems, and $(k - 2)$ of those of the S type.

In the nth region of the G type

$$\theta = \frac{\sin {}^1/_2\alpha}{\sin (n\alpha - {}^1/_2\alpha)} u_1 + \frac{\cos {}^1/_2 \alpha}{\sin (n\alpha - {}^1/_2\alpha)} u_2 + R_n \qquad (3.3)$$

In the S type regions contiguous with pistons P_1 and P_2 we have respectively

$$\theta = \frac{1}{\sin n\alpha} u_2 + R_{1n} \qquad (3.4)$$

$$\theta = \frac{\sin \alpha}{\sin n\alpha} u_1 + \frac{\cos \alpha}{\sin n\alpha} u_2 + R_{2n} \qquad (3.5)$$

$$R_n = \theta_0 + (V_1 + V_2) \left[\frac{(\sin {}^1/_2\alpha)^2 + 2 \cos {}^1/_2\alpha \sin {}^1/_2 n\alpha \sin [{}^1/_2 (n - 1)\alpha]}{2 \sin {}^1/_2\alpha \sin (n\alpha - {}^1/_2\alpha)} - \frac{2n - 1}{2}\right] \qquad (3.6)$$

$$R_{i1} = \theta_0 - V_i \qquad (3.7)$$

$$R_{in} = \theta_0 - V_i - \frac{4 (V_i \sin [{}^1/_2 (n - 2)\alpha] \cos [{}^1/_2 (n + 2)\alpha] + V_{3-i} \sin [{}^1/_2 (n - 1)\alpha] \cos [{}^1/_2 (n + 1)\alpha])}{\sin n\alpha}$$

$$(n \geqslant 2; \quad i = 1, 2) \qquad (3.8)$$

We shall prove the Theorem. At a certain distance from the pistons the Riemann waves

$$u_1 = -\cos\alpha\, (\theta - \theta_0), \qquad u_2 = \sin\alpha\, (\theta - \theta_0) \qquad (3.9)$$

$$u_1 = \theta - \theta_0, \qquad u_2 = 0 \qquad (3.10)$$

are generated which interact in regions of the G type bounded by charcteristics. Using (3.9) and (3.10) as the conditions along the characteristics $D'E'$ and $D_1'E'$ (Fig. 6) and the equation of the characteristic strip of (1.3), we obtain in the G_1 region

$$\theta = u_1 + \mathrm{ctg}\, {}^1/_2 \alpha u_2 + \theta_0$$

Area G_1 adjoins the regions of double waves of the S type through simple waves $D'P'$ and $D_1'P'$. Values of θ along the characteristics and the relationship between θ_1 and θ_2 defined by the condition of absence of flow through the wall which are required for the computation of these are known

$$\theta_1 = 0 \text{ in } D', \qquad \theta_1 \cos\alpha - \theta_2 \sin\alpha = 0 \text{ in } D_1'$$

Solutions of the mixed problem are of the form:

in region S_{11}

$$\theta = \csc\alpha\, u_2 + \theta_0 - V_1$$

in region S_{21}

$$\theta = u_1 + \mathrm{ctg}\,\alpha\, u_2 + \theta_2 - V_2$$

Hence for $n = 1$ the relationships (3.3)–(3.8) are fulfilled. By induction we readily conclude that Formulas (3.3)–(3.7) hold for any n.

If in a G type region the parabolic degenration

$$\theta_1^2 + \theta_2^2 = \csc^2(n\alpha - {}^1/_2\alpha) = 1, \quad \text{or} \quad n\alpha = {}^1/_2\pi + {}^1/_2\alpha$$

$$\alpha = \pi / (2n+1) = \pi / k, \qquad k = 2n+1, \quad n \geqslant 2$$

of Eq. (3.1) occurs, then region G_n degenerates into a line, the characteristics merge, and their slope coincides with that of the bisectrix of angle α

$$\frac{du_2}{du_1} = -\mathrm{tg}\, n\alpha = -\mathrm{tg}\left(\frac{\pi}{2} + \frac{\alpha}{2}\right) = \mathrm{ctg}\,\frac{\alpha}{2}$$

In the hodograph plane we have two regions of the S type joining along the bisectrix. In the physical plane the picture is completed by plotting the simple wave adjoining the region of constant motion, and extending up to the lines of piston intersection with the level lines perpendicular to the angle bisectrix.

If in the region of S type

$$\theta_1^2 + \theta_2^2 = 1 / \sin^2 n\alpha = 1, \qquad \alpha = \pi / 2n = \pi / k, \qquad k = 2n$$

then the S_{in} regions degenerate into lines coinciding with walls P_1 and P_2 respectively. In the hodograph plane the picture is completed by a G type region. In the physical plane we obtain two simple waves the level lines of which extend from the bisectrix perpendicularly to P_1 and P_2.

It is clear from geometrical considerations that when α does not satisfy (3.1) the characteristics G_n, or S_{in} extend beyond the boundaries of the natural flow region, and a solution free of singularities cannot be derived. The Theorem is proved.

4. At velocities V_1 and V_2 comparable in modulus to θ_0, but exceeding certain critical velocities, a vacuum zone appears in the flow. Problems of this kind were analyzed in detail in [1]. We shall establish a new family of such flows in the neighborhood of the "vacuum – gas" line. We shall namely prove that with certain natural restrictions the form of the "vacuum – gas" boundary is the same in both the hodograph and the $\xi_1\ \xi_2$ planes.

To establish this property it is sufficient in accordance with (1.3) to prove that $\lim \theta\theta_i = 0$ when $\theta \to \theta_i$ $(i = 1,2)$. This statement is obviously true when θ_i are finite for $\theta = 0$ (this obtains, e. g. in the case of compatible α and γ).

We shall assume that $\theta_i \to \infty$ when $\theta \to 0$. Let curve $u_1 = f(u_2)$ in the hodograph plane correspond to $\theta = 0$. We shall consider narrow strip L along this line in which $\theta \subset [0, \varepsilon]$ (ε is small), and in which lie curves $u_1 = \varphi(u_2, \xi)$ of θ constancy ($\theta = \varepsilon$), where φ is twice continuously differentiable with respect to u_2. We shall assume that in the corresponding strip L' in the $\xi_1\xi_2$ plane along the "vacuum-gas" curve $\theta\theta_i = F_i(u_2, \xi) \geqslant 0$ when $\theta = \xi$, where F_i are continuously differentiable with respect to u_2 ($\xi \in [0, \varepsilon]$), $\xi = 0$ corresponds to the "vacuum-gas" line $u_1 = f(u_2)$; on Fig. 6 this is curve AB, and in the $\xi_1\xi_2$ plane curve $A'B'$. The boundedness of F_i follows from the boundedness of flow region in the $\xi_1\xi_2$ plane.

We shall further assume that inequalities $\varphi' < 0$, $\varphi'' < 0$ are fulfilled in L (the prime denotes differentiation with respect to u_2), i. e. curves of θ constancy are convex upwards for any ξ . We shall furthermore assume that $f, f', f'', F_i(u_2, 0)$ and $\varphi, \varphi', \varphi''$, F_i differ correspondingly in L by the magnitude $O(\varepsilon)$.

Differentiating θ twice along curve $u_1 = \varphi(u_2, \xi)$, we obtain the conditions

$$\theta_1 + \varphi'\theta_2 = 0 \tag{4.1}$$

$$\theta_{11}\varphi'^2 + 2\theta_{12}\varphi' + \theta_{22} + \theta_1\varphi'' = 0 \tag{4.2}$$

After substitution of θ_{12} from (4. 2) we reduce Eq. (1. 2) with the help of (4. 1) to the form

$$^1/_2(\gamma - 1)\,\theta\,[\theta_{11} + \theta_{22} + \theta_1{}^3\varphi''] + {}^1/_2(\gamma - 3)\,\theta_1{}^2(1 + \varphi'^2) + 2 = 0 \tag{4.3}$$

Differentiating equality $\theta\theta_1 = F_1(u_2, \xi)$ along $u_1 = \varphi(u_2, \xi)$ and solving the derived relationship with respect to $\theta\theta_{11}$, we obtain with the aid of (4. 2)

$$\theta\theta_{11} = \frac{1}{\varphi'^3}\,[2\varphi'^2 F_1' + \theta\theta_{22}\varphi' - \theta\theta_2\varphi''] \tag{4.4}$$

Substituting $\theta\theta_{11}$ from (4.4) into (4. 3) and neglecting in (4. 3) magnitudes of the order of $O(\varepsilon)$ as compared with $O(1)$ and magnitudes of the order of $O(1)$ as compared with $\theta_1{}^2 + \theta_2{}^2$, we finally obtain the approximate equation to which (1. 2) is reduced in L

$$\theta\theta_{22} - A(u_2)\,\theta_2{}^2 + \frac{\gamma - 3}{\gamma - 1}\,\theta_2{}^2 = 0 \qquad \left(A(u_2) = \frac{F_2(u_2\ 0)\,f''}{f'(1 + f'^2)} \geqslant 0\right) \tag{4.5}$$

We fix a certain straight line $u_1 = c_1 = \text{const}$ and integrating along it Eq. (4. 5) from u_2 to $u_2{}^*$, where $(c_1, u_2{}^*)$ lies on curve $\theta = \varepsilon$ and (c_1, u_2) within L

$$\int\limits_{u_2}^{u_2^*} \frac{d\theta_2}{\theta_2} + \frac{\gamma - 3}{\gamma - 1} \int\limits_{u_2^*}^{u_2^*} \frac{d\theta}{\theta} - \int\limits_{u_2}^{u_2^*} A(u_2)\,\frac{d\theta}{\theta} = 0 \tag{4.6}$$

Applying to the last term of (4. 6) the generalized mean value theorem, and integrating we obtain $\ln\theta_2 + \dfrac{\gamma - 3}{\gamma - 1}\ln\theta - A^*\ln\theta = \ln C \quad \theta\theta_2 = C\theta^{A^* + 2/(\gamma - 1)} \to 0$, for $\theta \to 0$

because $A^* = A(u_2{}^0) \geqslant 0$ $(u_2 < u_2{}^0 < u_2{}^*)$, $C = \text{const}$. In this case clearly also $\theta\theta_1 \to 0$ when $\theta \to 0$, i. e. curve $\xi_i = u_i$ in the $\xi_1\xi_2$ plane corresponds to line $u_1 = f(u_2)$.

Critical velocities V^* (a vacuum zone does not occur when $V < V^*$) were calculated for the case of $V_1 = V_2 = V$ with $\alpha = {}^1/_2\pi$, $\gamma = 1.4$ and $\gamma = 2$ (in [1] $V^* = 0.42$ for $\alpha = {}^1/_2\pi$, $\gamma = 3$) was cited), and the obtained results are as follows:

$$V^* \approx 2.29 \qquad (\alpha = {}^1/_2\pi,\ \gamma = 1.4), \qquad V^* \approx 0.88 \qquad (\alpha = {}^1/_2\pi,\ \gamma = 2)$$

Note 4. 1. Along a vacuum line θ_i may be either finite, or infinite. However, as follows from (1. 2), already the second derivatives will necessarily tend to infinity for $\gamma \geqslant 3$.

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VARIATION OF GAS VELOCITY IN A NORMAL IONIZING SHOCK WAVE AND THE PROBLEM OF THE CONDUCTIVE PISTON

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The variation of gas velocity in an ionizing shock wave propagating along an initial magnetic field (a normal ionizing wave), when the gas magnetic viscosity is considerably greater than the remaining dissipative coefficients, is investigated in this paper. Obtained results are used for the derivation of solution of the problem of motion of a conductive piston. Similar investigations in which the wave front orientation with respect to the magnetic field was arbitrary, were the subject of paper [1].

An analysis of the limit case of normal ionizing shock waves is of interest in view of the numerous experimental investigations of such waves [2 and 3], and also due to the presence of a number of singularities in its solution as compared with the general case. Variations of the magnetic field profile, of density and other parameters in a supersonic normal ionizing shock wave and in the subsequent MHD rarefaction wave were computed in paper [4] in connection with the problem of discharge. It was assumed there that in a varying magnetic field the ionizing wave becomes an ionizing Jouguet wave. It will be shown in the following that this assumption is correct.

In the case of normal ionizing waves here considered the flow is a plane one, i. e. the gas velocity and the magnetic field lie in one and the same plane drawn through the normal to the wave front. We introduce the system of coordinates x, y, z with the x-axis directed along a normal to the wave, and the magnetic field component behind the wave $H_{z2} = 0$. Let in this coordinate system u and v be the velocity variations along the x- and y-axes.

In the case under consideration intermediate ionizing shock waves are absent, i. e. out of the five wave types [5] three only are possible, viz. supersonic fast, supersonic and subsonic slow ionizing waves. In supersonic ionizing waves the magnetic field, and consequently also the gas velocity tangent component do not vary. In the uv-plane points lying along the u-axis to the right of u' correspond to a fast wave.